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Neutrino Oscillations and the MSW effect in Random Solar Matter

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Abstract

We investigate the effects of random density fluctuations on neutrino oscillations in the Sun environment. We show how the average of certain quantities which can be used to describe the MSW effect can be computed analytically. We examine also the hypothesis commonly accepted that only perturbations inside the resonance layer can have relevance. The average amplitude, which gives the "coherent probability", is computed in an analytical and exact way for the case of colored δ-correlated Gaussian noise: the random perturbation induces a renormalization of the matter density which acquires an imaginary part proportional to the fluctuation magnitude in the resonance region. Integral equations are given for the density matrix of the system in the "optical" approximation.

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1 Introduction

In this work we investigate matter-enhanced neutrino flavor transformations, the MSW effect [1, 2], for the case of a exponentially decaying matter density with a colored Gaussian noise.

There are several examples of interesting transport and scattering processes induced or modified by the presence of a disordered or random medium: dislocations in crystals and the solid-liquid transition; random impurity potentials produce the localization of quantum wavefunctions which enables one to understand the transition between insulators and conductors; resistance anomalies at low temperature and in presence of magnetic fields of "weak localized" electron systems subject to a random potential ([3, 4]). The same basic localization phenomena explain also in optics the backscattering enhancement in electromagnetic wave scattering from a randomly rough surface ([5]). Usually, a strong dependence on the dimensionality of the problem is observed. In some other cases as in nuclear physics the consideration of random matrix hamiltonians allows the simplification of otherwise unmanageable systems.

For neutrino oscillations in presence of random or rapidly twisting magnetic fields considerable work has been done already [6, 7, 8, 9]. Random density fluctuations have received some attention only recently.

In [6], a differential equation for the averaged survival probability was derived for the case in which the random noise was taken to be a delta-correlated white Gaussian distribution. The differential equation was solved numerically and the neutrino evolution obtained. Arguments were given which indicate that if the correlation of the matter density fluctuations is small compared to the neutrino oscillation length at resonance, one obtains the same result as for the case of a delta-correlated noise. In [10] the more realistic case of colored noise was considered also in a numerical way and applied to Supernova dynamics. An approximate differential equation for the averaged survival probabilities was obtained using the hypothesis that the fluctuations should not affect the evolution far from the resonance. In [11] , the implications of random perturbations upon the Solar neutrino deficit are considered numerically. It is found that the MSW effect is rather stable under these fluctuations specially in the small mixing case but in anycase the experimental $(\Delta m^2, \cos\theta)$ exclusion curves get modified in an appreciable way.

In this work we try to develop analytical results for neutrino flavor oscillations induced by a random matter density. We are interested mainly in the persistence of the MSW effect. The obtention of concrete values for the survival probabilities and phenomenological consequences for the solar neutrino problem is left for a subsequent work. The obtention of these phenomenological consequences is not an easy task , at least, because the amplitude of possible density fluctuations in the Sun is poorly known experimentally and theoretically. Different arguments can give easily values for it differing by two orders of magnitude ([11]): between $0.1\% - 10\%$ of the local density. Stronger local inhomogeneities, for example at the near-surface dark spots should not be discarded.

Our basic starting point will be the exact analytical solution obtained in [12] for the neutrino oscillation amplitudes in presence of an exponentially decaying matter density.

The random component is consider as a perturbation to this solution. Inspired by the electromagnetic wave scattering in random media, *coherent* and *incoherent* transition probabilities are defined. The basic result of this work is that the coherent probability, which comes essentially from an averaged amplitude, can be computed exactly in some cases. Different integral equations are given for the incoherent probability using an "optical" approximation. Approximate solutions valid in a limited range are obtained for them. By the very nature of the procedure the results are valid or easi ly generalizable to any number of neutrino species.

The outline of this work is as follows. The first section, after the introduction of some known results, is dedicated to explore diverse quantities whose stochastic average can be computed exactly or at least by an easy approximation. One of these quantities is the determinant of the evolution operator of the system. We use a "naive" argument to show the plausibility that the influence of the random perturbation on the neutrino oscillations can be described by a complex redefinition of the matter density. This suggestion allows to define some *ansatz* probability. Another quantity is the "total cross section" of the system, we suggest the interest in further studying this quantity which can contain information on the presence and localization of the MSW resonance. In the same section, the physical supposition that only those random perturbations happening in the resonance layer can have importance is studied briefly . We argue that this supposition must be taken with care, because, as we show, even small phase shifts *before* the resonance region can have some appreciable importance in the final survival probability.

In the next section we define some perturbative expansion for the density matrix of the system. The *coherent* and *incoherent* parts of this matrix are defined. The coherent part being in some sense the zero-order approximation for the full density. Using an "optical" approximation, that is discarding a certain class of terms in the perturbative expansion, a very general integral equation for the averaged density is derived. In this integral equation, basic ingredients are both the coherent density and the averaged amplitude and the two point correlation of the matter density perturbation.

In Section (4) different particular cases are considered. A simpler expression for the previous integral equation is given for the case where the random perturbation is δ-correlated. A further simpler expression is given for the small mixing case of two neutrino species. In this case upper limits for the total averaged probability can be obtained depending on the coherent probability and an autocorrelation integral.

In Section (5) the coherent part is computed. In a first case the small mixing condition and different approximations are used. Averaging the amplitude amounts to the multiplication by a certain slowly time-varying diagonal matrix. In a second particular case, it is shown how the average can be computed exactly. The effect of the averaging is indeed a complex renormalization of the initial matter density as it was suggested earlier. Even if it is a very particular case, it is shown how it can have relevance in more realistic computations. Finally, making use of these averages, coherent survival probabilities and the "cross-sections" previously defined are computed.

2 Preliminary Considerations

2.1 The non-stochastic solution

The solution for the neutrino oscillations in solar matter described by the equation

$$i\partial_t \nu = (H_0 + \rho_0 \exp(-\lambda t) u A u^{-1})\nu \tag{1}$$

has been given in [12].

For $t \to \infty$, the solution can be written as (λ will be understood generally set to 1)

$$\nu(t) = \exp -iH_0 t \;\; U_r(\rho_0)\; \nu(0) \tag{2}$$

For any arbitrary time

$$\nu(t) = U(t, t_0)\nu(0) \tag{3}$$

with

$$U(t, t_0) = U_s(t)^{\dagger} U_s(t_0), \qquad U_s(t) = U_r\left(\rho_0 e^{-t}\right)\; \exp -iH_0 t \tag{4}$$

The solution can be extended to complex ρ_0, then U_s is not unitary but still keeps the same functional form, the previous expression must be changed (following the same argument used to derive Eq.(58) in [12]) to

$$U(t, t_0) = U_s(t)^{-1} U_s(t_0)$$

In the general case (ρ_0 real or complex) the coefficients of U_r in the Expression (2) are just confluent generalize hypergeometric functions of one order less than the dimension of the problem, with argument $-i\rho_0/\lambda$ and parameters which are combinations of the eigenvalues of H_0 and the elements of the mixing matrix u. For ρ_0 real, U_r becomes unitary and its coefficients adopt a form particularly simple and symmetric. As example, we write it explicitly for the two-dimensional case:

$$U_r(\rho_0) \;\; = \;\; \begin{pmatrix} F & \frac{V_{12}}{V_{11}}\, G \\ -\frac{V_{21}}{V_{11}}\, \exp(z)\; G^* & \exp z\; F^* \end{pmatrix} \tag{5}$$

with the shorthands

$$G = g(z) = \frac{V_{11}}{1+\beta} {}_1F_1(1 + V_{11}\; \beta, 2+\beta; z), \quad F = {}_1F_1(V_{11}\; \beta, \beta; z),$$

$$z = -i\rho_0, \quad \beta = \Delta m^2/2E$$

The algebraic properties of F and G guarantee automatically the unitarity of U_r. Its determinant is

$$\det U_r(\rho_0) = \exp -i\rho_0 \tag{6}$$

The survival probability is given by

$$P_{ee}(\rho_0, \beta, \theta) = \;\; 1 - 2S^2\theta \left(1 + C^2\theta \left\| {}_1F_1\left(i\beta C^2\theta, 1 + i\beta; -i\rho_0\right)\right\|^2\right) \tag{7}$$

or in the small mixing angle limit:

$$P_{ee}(\rho_0, \beta, \theta) = \;\; \left\| {}_1F_1\left(i\beta C^2\theta, i\beta; -i\rho_0\right)\right\|^2 \tag{8}$$

2.2 The average determinant

In this work we are interested to investigate what is the effect of introducing a random density perturbation, that means, when $\rho(t) = \rho_0 \exp(-\lambda t)$ is changed to $\rho(t)(1 + \delta(t))$ where δ is a stochastic Gaussian process of zero mean, characterized completely by a certain two point correlation function.

Having in mind the optical potentials and random matrix approximations used in nuclear physics [13] and wave scattering in random media [5], one could think that in our case the effect of such introduction should be approximately equivalent to a redefinition of the function $\rho(t)$ which in the most simple case would amount just to a renormalization of the constant ρ_0 (to a complex value in general in analogy with the complex wave numbers appearing in random media wave scattering).

We get some hint that such supposition is reasonable if we consider the average value of the determinant of the evolution operator of our differential equation. To compute that determinant and its average value is a trivial task.

The evolution operator can be expressed formally as a time-ordered (T) integral:

$$U(t, t_0) = T \;\exp -i \int_{t0}^{t} ds H(s) \tag{9}$$

Its determinant is simply the elementary exponential:

$$\det U = \exp -i \int_{t0}^{t} ds \mathrm{tr}\; H(s) \tag{10}$$

As $H(t)$ is assumed to be a Gaussian process, the statistical average is:

$$< \det U >=< \exp -i \int \mathrm{tr}\; H >= \exp -i \int < \mathrm{tr} H > -\frac{1}{2} \int\int << \mathrm{tr} H(t) \mathrm{tr} H(s) >> \tag{11}$$

Where $<< A\ B >>=< A\ B > - < A >< B >$.[1]

So, for the total hamiltonian H defined in Eq.(1) we have:

$$< \mathrm{tr}\; H(t) >= \Sigma + \rho_0 \exp -t \tag{12}$$

The constant term Σ is unimportant, it can be set to zero by a convenient redefinition of the zero energy. With

$$< \delta(t)\delta(s) >= k(t + s)g(\mid t - s \mid) \tag{13}$$

we have

$$<< \mathrm{tr} H(t)\ \mathrm{tr} H(s) >>= k(t + s)\rho_0^2 \exp -(t + s)\ g(\mid t - s \mid) \tag{14}$$

We are interested mainly in the limit $t \to \infty$, we take also $t_0 = 0$. After some elementary integrations we arrive at:

$$< \det U >= \exp -i \int_0^{\infty} dt\ \rho_0 \left(1 - \frac{1}{2}\rho_0 i k(t) \int_0^t g(u) du\right) \exp(-t) \tag{15}$$

[1] We will use indistinctly the notations $< A >$, $\overline{A}$, or $\mathcal{A}$ for the average of the variable A

We can suppose with generality that $g(u)$ goes to zero quickly for $u \to \infty$, that guarantees that we make a relatively small error if we substitute the upper limit of integration by ∞. For the case of $g(u)$ being a delta function, this approximation becomes exact. We will suppose also that the factor $k(t) \approx k$, constant.

So we can interpret that the introduction of the random perturbation induces a renormalization of the initial density which is proportional to the integral of the autocorrelation function.

$$\rho_0 \to \rho_{0r} = \rho_0 \left(1 - i\frac{k\rho_0}{2}\int_0^\infty g(u)du\right) \tag{16}$$

Under this approximation, the average determinant becomes:

$$< \det U >= \exp -i\rho_{0r} \tag{17}$$

the renormalization introduces an imaginary part that renders the operator $U(\rho_{or})$ non-unitary, on average; certainly the initial hamiltonian would become non hermitic if we would substitute directly it it ρ_o by ρ_{or}. Formally we can recover at least the condition $|< \det U >| = 1$ (but of course not the unitarity condition) adding to the initial hamiltonian a diagonal term of the form

$$H_r = i\rho_0^2 \exp -t \int_0^\infty dug(u) \quad I$$

I being the identity matrix. This amounts to a (complex) shift in the energy which is unobservable.

We note that for computing Eq.(7), the unitarity of U has been explicitly used in an important way ([12]). Our temptative *ansatz* is to suppose that the introduction of a stochastic term is equivalent to the consideration of a non-stochastic equation with a redefined initial maximal density, as obtained before, plus appropriated "counterterms" that render the full hamiltonian hermitic. The physical information, the transition probabilities, can be computed by analytical continuation of the original transition probabilities corresponding to the initial equation with the new redefined parameters.

So under this ansatz the averaged transition probability in presence of the random term is (for two dimensions, Eq.(7)) $P_{ee}(\rho_{or})$.

The rest of this article will be devoted essentially to a more rigorous justification of this prescription. We will show that at least for the, so called, *coherent* probability this assumption is true in a particular case.

2.3 Cross sections and the Optical Theorem.

As we will see later, while it is relatively easy to compute the average values $< U_{ij} >$, the averaged probabilities, which depend on quadratic quantities $<| U_{ij} |^2>$, are nearly inaccesibles analytically.

In fact, apart from survival probabilities, we are also simply interested to study whether the MSW effect survives or to which extent gets modified with the introduction of a

random perturbation. It would be important to find a quantity that both: gives us some information about the existence, amplitude or position in parameter space of the MSW effect and its stochastic average is easy to compute . In this sense, we propose to use the "scattering" matrix T defined by

$$U_I(t \to \infty) = 1 + T \tag{18}$$

where U_I is the evolution matrix defined before in some appropriate interaction representation.

T satisfies an "Optical Theorem":

$$\begin{aligned} U_I U_I^\dagger = 1 &= (1+T)(1+T^\dagger) && (19) \\ T + T^\dagger &= -TT^\dagger && (20) \end{aligned}$$

In particular we define the two quantities:

$$\begin{aligned} \sigma_1^i &= -2\Re T_{ii} = \sum_k \mid T_{ik} \mid^2 && (21) \\ \sigma_2 &= -2\Re \mathrm{tr} T = \sum_{ik} \mid T_{ik} \mid^2 \equiv \mid\mid T \mid\mid^2 && (22) \end{aligned}$$

We can identify the first expression with a total cross section in particle physics (sum over all final channels). The second is the sum of all total cross sections over initial channels. We expect that the quantities σ_1, σ_2 can give us interesting information about the scattering process, hopefully the MSW resonance should manifest itself in them. The important thing is that, because both depend linearly on T (U), their statistical average is rather easy to compute.

In the mass basis

$${}^M\sigma_1^i = -2\Re(U_{ii} - 1) = 2(1 - \Re U_{ii}) \tag{23}$$

In the weak basis $U_I = uU_r u^{-1}$, and we have a similar expression. The expression for σ_1^1 is particularly simple

$${}^W\sigma_1^1 = -2\Re((uU_r u^{-1})_{ii} - 1) = 2(1 - \Re \mathrm{tr} U_r V) \tag{24}$$

On the other hand, σ_2 is basis invariant and (d is the dimension of the problem)

$$\sigma_2 = 2(d - \Re \mathrm{tr} U_r) \tag{25}$$

In Fig.(1) we plot the quantities ${}^W\sigma_1^1, \sigma_2$ for particular parameters together with the ν_e survival probability. We see that both show a prominent peak in the resonance region. They reproduce the secondary extrema as well .

For $\beta \to 0$ or $\beta \to \infty$, $\sigma_2 \approx 0$ this implies

$$2 \approx \Re \mathrm{tr} U_r \equiv (1 + \cos\rho_0)\Re F + \sin\rho_0 \Im F \tag{26}$$

For the value of ρ_0 used in the plot: $1 + \cos\rho_0 \approx 2, \sin\rho_0 \approx 1/5$ (a small but finite value). We know also that $P_{ee} \simeq \mid F \mid^2 \leq 1$, then $\Re F \sim 1$ and $P_{ee} \sim 1$.

On the other hand, for $\beta \to \beta_{res}$, $\sigma_2 = 4, {}^W\sigma_1^1 = 2$. $\Re \mathrm{tr} U_r = 0, \Re \mathrm{tr} U_r V = 0$ imply $\Im F, \Re F \approx 0$ and we obtain the resonance probability $P_{ee} \approx 0$.

Later we will show the same type of plot for the averaged quantities.

2.4 The influence of the resonance layer

From a physical point of view, and, as is usually assumed, the only perturbations that can have an influence on the final transition probabilities are those which happen inside the resonance layer.

We have supposed Gaussian random perturbations; there is a finite probability that in any moment through the neutrino path, a strong local fluctuation makes the density term similar to the difference ΔE provoking in this way a resonant level crossing. Numerical studies ([11, 6, 10]) seem to show that this possibility in fact doesn't happen easily at least for small mixing angles.

But, we want to show that small changes of the neutrino wave function *before* arriving to the resonance region can have an important effect on the final probability.

Let's suppose that for any circumstance the flavor components of the neutrino wave function acquires a relative phase ϕ at some t_1 much later than its creation time. At infinite its wave function would be

$$\begin{aligned} \nu(t \to \infty) &= U(t \to \infty, t_1) \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix} U(t_1, t_0)\nu_0 \\ &= U_s^\dagger(\infty)\ \Sigma\ U_s(t_0)\nu_0 \end{aligned} \tag{27}$$

Where the matrix Σ is defined by the above equation.

In Section (5.1) we will see that, although the similarity is not complete, indeed the effect of random perturbations can be accounted for by the insertion of a certain matrix inside the non-random U.

In Fig.(2) we show the survival probability as a function of the intermediate time t_1 (or r/r_0 in the figure) and ϕ. We see that the phase shifts introduced after the resonance layer have no or very little effect. The shifts introduced at the beginning or very clearly before the resonance region can have a drastic influence on the final probability. The obvious conclusion from this is that if random perturbations affect (even slightly) the phase of the wave function long before the resonance region, then they can have an appreciable effect on the final survival probability.

3 Formulation of the Main Approach.

The density operator of any system defined by a hamiltonian $H(t) = H_0 + W(t)$ and certain initial conditions, is given, in terms of U, the evolution operator, by:

$$\rho(t) = U(t, t_0)\rho(t_0)U(t, t_0)^\dagger \tag{28}$$

If the potential vanishes for $t \to \infty$, the asymptotic propagator obeys a free Schroedinger equation and can be cast in the form:

$$\rho(t \to \infty) = \exp -iH_0 t \ \rho_{as} \ \exp iH_0 t \tag{29}$$

ρ_{as} is a time-independent operator to be determined for any particular problem and initial conditions.

Our objective will be to derive an integral equation for the statistical average of the density operator $\overline{\rho(t)}$ in the $t \to \infty$ limit for the special type of stochastic potential we have discussed in the previous section.

Using perturbation theory around the known solution of the non-stochastic part of $W(t)$, $V^0(t) \equiv < W(t) >= \rho_0 \exp(-t)V$, $V^2 = V$ we have for U:

$$U(t, t_0) = U^0(t, t_0) + \sum_n U^{(n)}(t, t_0) \tag{30}$$

As we have seen before, for this kind of potential we can factorize the operator U^0 in "creation" and "anhilation" operators:

$$U^0(t, t_0) = U_s^{0\dagger}(t) U_s^0(t_0) \tag{31}$$

so

$$U(t, t_0) = U_s^{0\dagger}(t) \Sigma(t, t_0) U_s^0(t_0) \tag{32}$$

$$\Sigma(t, t_0) = 1 + \sum_{n=1}^{\infty} \int_{t0}^{t} d\tau \ V_I(\tau_n) \ldots V_I(\tau_1) \tag{33}$$

$$(d\tau = d\tau_1 \times \ldots \times d\tau_n)$$

$$\tag{34}$$

where

$$V_I(t) = U_s^0(t) \left(-iV(t)\right) U_s^{0\dagger}(t), \quad V(t) = W(t) - V^0(t) \tag{35}$$

We define equally the density matrix in the interaction representation

$$\rho_I(t) = U_s^0 \rho U_s^{0\dagger} = \Sigma(t, t_0) \rho_I(t_0) \Sigma(t, t_0)^\dagger \tag{36}$$

In the limit $t \to \infty$, $U_s^0(t) \to \exp iH_0 t$ and we identify

$$\rho_{as} = \rho_I(t \to \infty) = \Sigma(\infty, t_0) \ \rho_I(t_0) \ \Sigma(\infty, t_0)^\dagger \tag{37}$$

The average density operator contain a contribution from neutrinos which have scattered coherently (or specularly in the electromagnetic wave analogy). This contribution is obtained averaging the amplitude U:

$$< \rho^{coh} >=< U > \rho_0 < U^\dagger > \tag{38}$$

The contribution from the incoherent, or diffuse, component is obtained by subtraction.

$$< \rho^{incoh} >=< \rho > - < \rho^{coh} > \tag{39}$$

All the "randomness" information is included in the operator Σ, so the Eq.(36) holds also for the respective statistical averages: $< \rho >_I =< \rho_I >$.

Taking[2] for the initial density matrix

$$\rho_0 = u(1, 0, \ldots, 0)u^{-1} = V$$

the macroscopical $\nu_e \nu_e$ transition probability is,

$$P_{ee}^M(t) = \mathrm{tr} < \rho(t) > V \tag{40}$$

Taking $t \to \infty$ and averaging out oscillatory terms

$$P_{ee}^M = \Sigma_i \rho_{as,ii} V_{ii} \quad (\simeq \rho_{as,11} \text{ for small mixing: } V_{11} \to 1) \tag{41}$$

From Eq.(36), the statistical average of ρ_I can be decomposed in a sum of terms of the form:

$$\begin{aligned} \overline{\rho_I} &= \sum_{p,q} A_{pq} \\ A_{pq} &= < \int V_I \overset{p}{\cdots} V_I \;\rho_{0I} \int V_I^\dagger \overset{q}{\cdots} V_I^\dagger > \end{aligned} \tag{42}$$

It is assumed that the stochastic potential is a Gaussian process with zero mean. This implies that the average of terms with an odd number of V's is zero. On the other hand the average of the terms with a even number of V's can be decomposed in a sum of two-V averages extended over all possible combinations. For example

$$\overline{VVVV} = \overline{VV}\;\overline{VV} + \overset{1}{\mathcal{V}}\;\overline{VV}\;\overset{1}{\mathcal{V}} + \overset{1}{\mathcal{V}}\;\overset{2}{\mathcal{V}}\;\overset{1}{\mathcal{V}}\;\overset{2}{\mathcal{V}} \tag{43}$$

The numbers over the *calligraphic V's* identifies the pairs which are averaged.

According to arguments developed in [13, 6] the terms which contain "cross" averages, for example the last term in Expression (43), can be discarded to a good approximation for pair correlation functions which are significantly non-zero only for a relatively short time difference (for example for delta-functions). We will suppose that this is always true in our case; in the particular case of the coherent part we will not need this approximation at all.

For convenience we will use from now on a "two time" density matrix, the statistical average of ρ_I is given by

$$\overline{\rho_I(t,s)} \;\;=\;\; \overline{\rho_I^{coh}} +$$

[2] There should not be risk of confusion between ρ, ρ_0 as density matrices or matter densities.

$$+\sum_{p,q,r,s}\int\overline{V\overset{p}{\cdots}V}\overset{1}{\mathcal{V}}\overline{V\overset{q}{\cdots}V}\rho_{0I}\int\overline{V\overset{r}{\cdots}V}^{\dagger}\overset{1}{\mathcal{V}}^{\dagger}\overline{V\overset{s}{\cdots}V}^{\dagger}+$$
$$+\sum\int\overline{V\ldots V}\overset{1}{\mathcal{V}}\overline{V\ldots V}\overset{2}{\mathcal{V}}\overline{V\ldots V}\rho_{0I}\int\overline{V\ldots V}^{\dagger}\overset{2}{\mathcal{V}}^{\dagger}\overline{V\cdots V}^{\dagger}\overset{1}{\mathcal{V}}^{\dagger}\overline{V\cdots V}^{\dagger}+$$
$$+\cdots \tag{44}$$

The first summand is the coherent part and its computation amounts to sum the series

$$<1+\int V_I+\int V_I\int V_I+\ldots>=<T\exp\int V_I>\equiv\overline{\Sigma} \tag{45}$$

Note that the average can be performed exactly without neglecting "crossed" pair averages, so the last relation Eq.(45) can be considered exact. Interchanging averaging and time ordering:

$$<T\exp\int_{t0}^{t}V_I>=T<\exp\int_{t0}^{t}V_I>=T\exp\frac{1}{2}\int_{t0}^{t}\int_{t0}^{t}<<V_I(s_1)V_I(s_2)>>ds_1ds_2 \tag{46}$$

From this expression an effective potential V_I^{eff} can be defined which summarizes the influence of the stochastic potential and such that

$$\Sigma^{eff} \equiv T\exp\int_{t0}^{t}V_I^{eff}(t')dt'=\overline{\Sigma} \tag{47}$$

$$V_I^{eff}(t') \equiv \frac{1}{2}\int_{t0}^{t}<<V_I(t')V_I(s)>>ds \tag{48}$$

We can derive an alternative approximate expression ignoring crossing terms in Eq.(45). The propagator computed in such a case will be called the optical propagator in analogy with nuclear physics. It is easy to check that the optical propagator satisfies the equations

$$\Sigma^{opt}(t,t_0)=1+\int_{t0}^{t}d\tau\ V_I^{opt}(\tau)\Sigma^{opt}(\tau,t_0) \tag{49}$$

$$V^{opt}(\tau)=\int_{t0}^{t}d\tau_2\ \overset{1}{\mathcal{V}}_I(\tau)\Sigma^{opt}(t,\tau_2)\overset{1}{\mathcal{V}}_I(\tau_2) \tag{50}$$

In general V_I^{opt},Σ^{opt} are not neccesarily the same as the V_I^{eff},Σ^{eff} defined previously. Also it is not neccesarily more difficult to compute the former than the latter. In the important particular case of a δ-correlated potential both coincide.

We can compute all the further terms in the expansion of $\overline{\rho_I}$ (Eq.(44)) making use of the following expression; for an arbitrary non-stochastic operator K, the sum of the series:

$$S(t,s)=<K(t,s)+\int_{t0}^{t}V(\tau)K(\tau,s)+\int_{t0}^{t}V(\tau_1)\int_{t0}^{\tau 1}V(\tau_2)K(\tau_2,s)+\ldots> \tag{51}$$

is equivalent to solve the integral equation

$$S(t,s) = K(t,s) + \int_{t0}^{t} d\tau\ V(\tau)S(\tau,s). \tag{52}$$

and to apply the statistical average. The solution is given by

$$S(t,s) =< T\exp\int_{t0}^{t} V \times \left[K(0,s) + \int_{t0}^{t} d\tau\ \left(T\exp\int_{t0}^{\tau} V\right)^{-1} > \partial_\tau K(\tau,s)\right] \tag{53}$$

which, using the properties of the time ordered exponential and putting $K(0,s) = 0$, is equal to

$$S(t,t_0) = \int_{t0}^{t} d\tau < T\exp\int_{\tau}^{t} V > \partial_\tau K(\tau,s) \tag{54}$$

Using this formula iteratively in Eq.(44) we obtain the expression:

$$\begin{aligned}\overline{\rho_I^{incoh}(t,s)} \quad &= \quad \int_{t0}^{t} \Sigma^{eff}(t,\tau)\overset{1}{\mathcal{V}}_I(\tau)\Sigma^{eff}(\tau,t_0)\rho_{0I}\left(\int_{t0}^{s}\cdots\right)^\dagger \\ &+ \int_{t0}^{t} \Sigma^{eff}(t,\tau)\overset{2}{\mathcal{V}}_I(\tau)\int_{t0}^{\tau} \Sigma^{eff}(\tau,\tau_2)\overset{1}{\mathcal{V}}_I(\tau_2)\Sigma^{eff}(\tau_2,t_0)\rho_{0I}\left(\int_{t0}^{s}\cdots\right)^\dagger \\ &+ \cdots \end{aligned}\tag{55}$$

The operators to the right sides of ρ_{0I} are equal to those in the left sides, changing the limit of integration and taking the hermitic conjugate.

As it can be seen by explicit developing, the series given by Eq.(55) is equivalent to the following integral equation for the total density ρ_I:

$$\overline{\rho_I(t,s)} = \Sigma^{eff}(t,t_0)\rho_{0I}\Sigma^{eff\dagger}(s,t_0) + \int_{t0}^{t}\int_{t0}^{s} \Sigma^{eff}(t,\tau_1)\overset{1}{\mathcal{V}}_I(\tau_1)\ \overline{\rho_I(\tau_1,\tau_2)}\ \overset{1}{\mathcal{V}}{}^\dagger_I(\tau_2)\Sigma^{eff\dagger}(s,\tau_2) \tag{56}$$

and, at least formally, we have solved the problem in the approximation which neglects "crossed" terms.

4 The Integral Equation for δ-correlated noise.

Let's suppose that the two point correlation matrix has the form:

$$< V_I(t)V_I(s) >= -ik(t)\rho(t)\delta(t-s)V_I(t) \tag{57}$$

or more generally, for any non-stochastic operator $K(t,s)$,

$$< V_I(t)\ K(t,s)\ V_I(s) > (\equiv \overset{1}{\mathcal{V}}K\overset{1}{\mathcal{V}}) = k(t)\delta(t-s)V_I(t)\ K(t,t)\ V_I(t) \tag{58}$$

The integral equation for the operator density becomes (Eq.(56))

$$\overline{\rho_I(t)} = \Sigma^{eff}(t,t_0)\rho_{0I}\Sigma^{eff\dagger}(t,t_0) + \int_{t0}^{t} d\tau k(\tau)\Sigma^{eff}(t,\tau)V_I(\tau)\overline{\rho_I(\tau)}V_I^{\dagger}(\tau)\Sigma^{eff\dagger}(t,\tau) \quad (59)$$

Or recalling the definition for the representation "I", Eqs.(35-36):

$$\overline{\rho(t)} = \overline{\rho^{coh}(t)} + \int_{t0}^{t} d\tau\ k(\tau)\ U(t,\tau)V(\tau)\overline{\rho(\tau)}V^{\dagger}(\tau)U^{\dagger}(t,\tau) \quad (60)$$

The probability P_{ee}^{M} of ν_e survival is computed using Eqs.(40-41). We have the expression

$$P_{ee}^{M}(t) = P_{ee}^{coh}(t) + \int_{0}^{t} d\tau\ k(\tau)\rho^2(\tau)\ \mathrm{tr}U(t,\tau)V\overline{\rho(\tau)}VU^{\dagger}(t,\tau)V \quad (61)$$

Using the property $V^2 = V$, this expression can be written also in the form

$$P_{ee}^{M}(t) = P_{ee}^{coh}(t) + \int_{0}^{t} d\tau\ k(\tau)\rho(\tau)^2\ \mathrm{tr}U_R(t,\tau)\overline{\rho_R(\tau)}U_R^{\dagger}(t,\tau) \quad (62)$$

Where $X_R = VXV$. At this point further approximations are neccesary in order to simplify the expressions in eqs.(61-62)..

4.1 The small mixing approximation.

In the small mixing, two dimensional case, i.e. taking the limit

$$V, \rho_0 \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (63)$$

such that $X_R = VXV \to X_{11}\rho_0$ the integral which appears in Eqs.(61-62) becomes

$$\int_{0}^{t} \rho^2(\tau)k(\tau) \mid U_{11}(t,\tau) \mid^2 \overline{\rho_{11}(\tau)} \quad (64)$$

In this limit, the weak and mass basis coincide and

$$\begin{array}{ccc} P_{ee}^{M} & \to & \overline{\rho_{11}} \\ P_{ee}^{coh} & \to & \mid U_{11} \mid^2 \end{array}$$

So

$$P_{ee}^{M}(t) = P_{ee}^{coh}(t) + \int_{0}^{t} d\tau\ k(\tau)\rho^2(\tau)P_{ee}^{coh}(t,\tau)P_{ee}^{M}(\tau) \quad (65)$$

Eq.(65) becomes exact for vanishing mixing angles.

A similar reasoning can be used to derive an equivalent expression for $P_{e\mu}^{M}$: in this case

$$P_{e\mu}^{M} = \mathrm{tr}\rho W$$

W is a certain matrix such that $W^2 = 0,\ W \cdot V = 0$. The final result is

$$P^M_{e\mu}(t) = P^{coh}_{e\mu}(t) + \int_0^t d\tau\ k(\tau)\rho^2(\tau) P^{coh}_{e\mu}(t,\tau) P^M_{ee}(\tau) \tag{66}$$

Defining total probabilities

$$P^{M,coh} = P^{M,coh}_{ee} + P^{M,coh}_{e\mu}$$

we obtain

$$P^M(t) = P^{coh}(t) + \int_0^t d\tau\ k(\tau)\rho^2(\tau) P^{coh}(t,\tau) P^M_{ee}(\tau) \tag{67}$$

This equation is specially useful to set an upper limit on P^M or alternatively on the magnitude of the "crossed" terms that we discarded.

We obtain the strict limit

$$P^M(t) \leq P^{coh}(t) + \int_0^t k(t)\rho(t) \tag{68}$$

Setting $k(\tau) \leq k \exp \beta t$ ($\beta < 2$, eventually $\beta = 0$) and taking $t \to \infty$ it follows

$$P^M \leq P^{coh} + \frac{k\rho_0^2}{2-\beta} \tag{69}$$

In principle $P^M \equiv 1$, a departure from this value signals a breakdown of the validity of the "optical" approximation.

If we denote by "C" the remaining "crossed" terms not incorporated in the optical approximation then

$$1 = P^M = P^{coh} + \left[\int \cdots\right] + C \tag{70}$$

So

$$1 - P^{coh} - \frac{k\rho_0^2}{2-\beta} \leq C \tag{71}$$

An actual estimate can be obtained if the function $k(\tau)\rho^2(\tau)$ is of type exponential, so only the values for $\tau \to 0$ of the integrand will contribute significantly. We take the approximation

$$P^{coh}(t,\tau)P^M_{ee}(\tau) \simeq P^{coh}(t,0)P^M_{ee}(\epsilon),$$

with ϵ some small number. Under this approximation:

$$P^M(t) = P^{coh}(t)\left(1 + k\rho_0^2 P^M_{ee}(\epsilon)\mu(t)\right) \tag{72}$$

$P^M_{ee}(\epsilon)$ should be typically very small and not very dependent on other parameters; Note that for $t \to \infty$ $\mu(t)$ only varies between $\approx 0.5 - 1$ irrespective of the exponential behavior of $k(t)$.

We see that the coherent density, or coherent probability, plays an important role not only in its own right but also in the computation of the global density or probability.

5 Computation of the Coherent probability.

The effective propagator defined in Eq.(47) is

$$\Sigma^{eff}(t,t_0) = T\exp\left(\frac{-i}{2}\right)\int_{t0}^{t} dt' k(t')\rho(t')V_I(t') \tag{73}$$

So the effective potential becomes

$$V_I^{eff}(t) = \left(\frac{-i\rho(t)k(t)}{2}\right)V_I(t) \tag{74}$$

In the same case, optical propagator reads:

$$\begin{aligned}\Sigma^{opt}(t,t_0) &= 1+\int_{t0}^{t} d\tau_1\int_{t0}^{\tau 1} d\tau_2 \overset{1}{\mathcal{V}}_I(\tau_1)\Sigma^{opt}(\tau_1,\tau_2)\overset{1}{\mathcal{V}}_I(\tau_2)\Sigma^{opt}(\tau_2,t_0)\\ &= 1+\frac{-i}{2}\int_{t0}^{t} d\tau\ k(\tau)\rho(\tau)V_I(\tau)\Sigma^{opt}(\tau,t_0)\end{aligned} \tag{75}$$

The last relation in Eq.(75) is just the integral equation for Σ^{eff} which proves that both are identical in this case.

Here:

$$V^{eff}(t) = \frac{-i}{2}\rho_0^2\exp(-2t)k(t)V \tag{76}$$

For any function $k(t)$, a possibility is to compute Σ^{eff} by perturbation theory, considering the addition of a "small" term $V^{eff}(t)$ to the hamiltonian $H(t)$ of which we know already the exact solution. In Section 6 we will follow this procedure for a particular $k(t)$. In fact we will be able to sum the full perturbative series.

Another possibility, somehow less systematic, will be explored in the next section where we will consider the small mixing limit.

5.1 Small Mixing

Recalling the definition of U_s in Eq.(4) and inserting it in Eq.(76) we get

$$V_I^{eff}(t) = \frac{-\rho_0^2}{2}\exp(-2t)k(t)\begin{pmatrix} \mid U_{s11}\mid^2 & U_{s11}U_{s12}^* \\ U_{s21}U_{s11}^* & \mid U_{s21}\mid^2\end{pmatrix} \tag{77}$$

In the previous matrix we will further suppose we can neglect the off diagonal terms containing oscillating terms of the type $\exp(iEt)$.

$$V_I^{eff}(t) \simeq \frac{-\rho_0^2}{2}\exp(-2t)k(t)\begin{pmatrix} \mid U_{r,11}\mid^2 & 0 \\ 0 & 1-\mid U_{r,11}\mid^2\end{pmatrix} \tag{78}$$

The time ordered integral is then trivial to perform, taking $t\to\infty$ we obtain

$$\Sigma^{eff}(\infty,t_0) = \begin{pmatrix} \exp(-\frac{\rho_0^2}{2}\phi_1) & 0 \\ 0 & \exp(\frac{\rho_0^2}{2}(\phi_1-\phi_2))\end{pmatrix} \tag{79}$$

where

$$\phi_1(\rho_0, t_0) = \int_0^{\rho(t_0)/\rho_0} dxxk(\log(x)) \mid U_{r,11}(\rho_0 x) \mid^2 \tag{80}$$

$$\phi_2 = \int_0^{\rho(t_0)/\rho_0} dxxk(\log(x)) \tag{81}$$

For $k(\log(x)) \sim kx^\beta$, both quantities ϕ_1, ϕ_2 are small, of order $O(1)$. For an arbitrary time t, the expressions will be analogous but changing the inferior limit of integration into $\epsilon \approx 0$. So we expect only a very slow dependence on t, this can be seen also from the presence of a factor $\exp(-2t)$ in Eq.(78).

In the small mixing limit $\mid U_{r,11} \mid^2$ is the survival probability P_{ee} in the absence of random perturbation (Eq.(8)). In a typical interesting case (presence of resonance) $P_{ee} \equiv P_1 \approx 1$, for $\rho_0 e^{-t_0} > \rho_{res}$ and small ($P_0 \approx 0$) for $\rho_0 e^{-t_0} < \rho_{res}$. In the absence of resonance (formally $\rho_{res} \to \infty$) $P_{ee} \approx 1$ for all ρ_0, t_0. Using this behavior we can estimate $\Sigma^{eff}(t \to \infty, t_0 = 0)$:

$$\Sigma^{eff}(\rho_0 > \rho_{res}) = \begin{pmatrix} \exp\left[\frac{-\rho_{res}^2 k}{4}(P_1 - P_0) - \frac{\rho_0^2 k}{4} P_0\right] & 0 \\ 0 & \exp\left[\frac{\rho_{res}^2 k}{4}(P_1 - P_0) + \frac{\rho_0^2 k}{4}(P_0 - 1)\right] \end{pmatrix}$$

$$\Sigma^{eff}(\rho_0 < \rho_{res}) = \begin{pmatrix} \exp\frac{-\rho_0^2 k}{4} P_1 & 0 \\ 0 & \exp\frac{-\rho_0^2 k}{4}(P_1 - 1) \end{pmatrix} \tag{82}$$

The coherent part of the macroscopic probability is (Eqs.(40-41))

$$P_{ee}^{M,coh} = \Sigma_i \left(\Sigma^{eff} \rho_{0I} \Sigma^{eff}\right)_{ii} V_{ii} \tag{83}$$

For a $\Sigma^{eff} = diag(A, B)$ as before, we have

$$P_{ee}^{M,coh} = A^2 \mid U_{r,11}^0 \mid^2 V_{11} + B^2 \mid U_{r,12}^0 \mid^2 V_{22} \tag{84}$$

6 The special case k(t)=exp t.

6.1 The Coherent Probability

It is interesting to consider the case where $k(t) = k \exp t$. Then it is possible to further continue the exact analytical expressions. Eq.(57) becomes:

$$< V_I(t) V_I(s) >= -ik\rho_0 \delta(t - s) V_I(t) \tag{85}$$

and the effective potential is

$$V_I^{eff} = \left(\frac{-ik\rho_0}{2}\right) V_I; \quad \text{or } V^{eff}(\rho_0) = V\left(\frac{-ik\rho_0^2}{2}\right) \tag{86}$$

The effective potential has the same functional form as the original potential $\sim \exp -t$ but with the constant ρ_0 redefined. Σ^{eff} can be computed exactly in a simple way.New operators U_s, U_r, U are defined by:

$$U_s^{0\dagger}(t)\Sigma^{eff}(t,t_0)U_s^0(t_0) = U_s^{-1}(t)U_s(t_0) \equiv U(t,t_0) \tag{87}$$

The new operators are just a renormalized version of the old ones U_s^0, U^0.

$$U(t,t_0;\rho_0) = U^0\left[t,t_0;\rho_0\ \left(1-\frac{ik\rho_0}{2}\right)\right] = \exp iH_0t\ U_r\left[\rho_0(1-\frac{ik\rho_0}{2})e^{-t}\right] \tag{88}$$

In the limit $t \to \infty$, both U_s^0, U_s, which depend only on the free hamiltonian H_0, are identical, so we get the relation

$$\begin{aligned}\Sigma^{eff}(t\to\infty,t_0) &= U_s(t_0)U_s^{0\dagger}(t_0) && (89)\\ &= U_r(\rho_{or})U_r^{\dagger}(\rho_0) \quad (t_0=0) && (90)\end{aligned}$$

The asymptotic density (Eqs.(29-37)) is given by

$$\rho_{as}^{coh} = U_s\rho_0U_s^{\dagger} \tag{91}$$

The coherent probability can be computed using Eq.(40) which amounts essentially to redefine ρ_0 in Eq.(7).

The total incoherent part

$$P^{M,in} = 1 - \left(\mid U_{r,11}\mid^2 + \mid U_{r,12}\mid^2\right) \tag{92}$$

is not equal to zero because the matrices U_s, U_r are not unitary now.

7 Some numerical results

We see that the choice $k(t) \sim k_1 \exp t$ allows for a fully analytical exact computation of the coherent probability. There are no physical grounds for this choice, as there are no physical grounds for any other selection, for example $k(t) \sim k_2$), as long as we don't have a very detailed knowledge of the Sun structure. If we suppose that random fluctuations are only important if they happen in the resonance layer (but see the warning commentary in a previous section) the result of both choices should be approximately equivalent if we take k_1, k_2 such that

$$\begin{aligned} k_1\exp t_{res} &\approx k_2 && (93)\\ \text{or} \qquad k_1\rho_0 &\approx k_2\rho_{res} && (94)\end{aligned}$$

where ρ_{res}, t_{res} are the position and local density of the resonance layer.

Outside the resonance region, for $t \to \infty$, $k_1 \exp t >> k_2$ but in anycase the products $k_1 \exp t\rho^2(t), k_2\rho^2(t) \to 0$.

For $t_{res} > t \to 0$, $k_1 \exp t = \exp(t - t_{res}) k_2$; for a distance $(t - t_{res})$ equivalent to a quarter of the solar radius that means $k_1 \exp t \approx k_2/3$, for half of the solar radius $k_1 \exp t \approx k_2/10$. The differences between both cases are not so strong as it could be thought due to the presence of the extra exponential.

In Fig.(3) we plot the averaged coherent survival probability P_{ee}^{coh} and the total coherent probability P^{coh}. In the top figure we make use of the Expression (82) for Σ^{eff}. In the bottom one we plot the raw Formula (88). In both figures the probability is strongly suppressed for neutrinos created near the origin. In the bottom one, the equivalent fluctuation level at the resonance region varies for different creation point r/r_0, but, in spite of the exponential behavior of $k(t)$, this variation is rather modest. For $k = 10^{-4}$ as used, the equivalent fluctuation at the resonance region $r/r_0 \approx 0.5 - 0.6$ is only $\approx 7\%$ of the local density for a neutrino created at $r/r_0 = 1/10$ and $\approx 13\%$ for a neutrino created at the center.

In Fig.(4) we plot instead the averaged coherent probabilities computed using the corrected formula $\rho_{or} = \rho_0(1 - ik\rho_{res}/2)$ which guarantees at least uniformity at the resonance region for different values of k.

For small mixing angles, P_{ee}^{coh} shows a moderate damping as k increases. The variation for the total coherent probability is much stronger. Beyond the resonance region, practically all $P^{coh} \approx P_{ee}^{coh}$.

As the mixing angle increases the pattern is different, P^{coh} practically doesn't suffer alteration but P_{ee}^{coh} is strongly diminished.

For both angles: for $k = 10^{-3}, 10^{-2}$, P^{coh} is practically zero in the inner creation regions. Here nearly all the eventual survival probability must come from the diffuse, incoherent scattering.

It is also shown the P^M defined by Eq.(72). For illustration purposes we choose the maximal value $P_{ee}^M(\epsilon) = 1$. Even in this maximal case P^M is far from unity, this is particularly evident for the strongest fluctuations. For large regions the "optical" approximation hardly differs from the coherent probability and fails to be an appreciable improvement. The lower limit for the "crossed" or "non-optical" corrections , although without much value numerically, can be interpreted as indicative that it is precisely around the resonance region where the "optical" approximation failure is strongest.

The average of the cross sections defined by Eq.(24-25) is obtained inserting the average value of the matrix U, U^{eff}, computed using either Σ^{eff} from Eq.(82) (Fig.(5), top) or $U_r(\rho_{or})$ (the two bottom figures). In both cases the approximated expression $\rho_{res} \simeq \beta \cos 2\theta$ has been used. For the smaller $k \approx 10^{-4}$ (or 1% fluctuations) the effect of the random perturbation is nearly negligible in both cases. The discrepancy in the large β behavior has the same origin as the difference between Fig.(3) (top) and Fig.(4). For $k \approx 10^{-3}, 10^{-2}$ the effect can be appreciable. The interpretation of these plots in terms of concrete survival probabilities is problematic and subject to further study, but what it is clear already from them is that the basic behavior, the very existence, position and width of a peak corresponding to the resonance layer remains unaltered. There are no discrepancies in this region between the two approaches.

In Fig.(6), we plot the averaged survival probability for different k using the *ansatz* formula for $P_{ee}(\rho_{or})$ of Section 2.2 and $\rho_{or} = \rho_0(1 - i\rho_{res}k/2$. There is consistency between the behavior expressed here and that of P_{ee}^{coh}, P^{coh}. If we take at face-value this plot and we compare it with Fig.(3) we arrive at the conclusion that for neutrinos created much before the resonate region the total survival probability comes essentially from the large avalaible quantity of "incoherent" probability. As happens in other physical circumstances, the presence of "diffuse" scattering disfavors transitions $\nu_e \to \nu_\mu$ and produces an enhancement of P_{ee} (state localization). In the same figure, the quick increase of the effect of the random perturbations with k is somehow surprising; for $\approx 10\%$ fluctuations the effect is already quite considerable ("reverse" MSW effect in the case of large mixing angle).

8 Brief summary of results and further conclusions.

The most important result of this work is the derivation of analytical exact expressions for the average *coherent* transition probability for a special case of colored δ-correlated Gaussian noise. We have shown that in this case the consequence of the presence of the noise is a complex renormalization of the matter density. Other approximative expressions for more general cases have been developed as well. It has been suggested an enhancement of the survival probability when the *incoherent* probability becomes dominant.

It has been proposed for the first time to consider new scattering "cross sections". It has been shown how they are able to describe themselves the MSW effect; clearly further work has to be done in this respect. The main importance of these quantities is that their statistical average is computable in a simple way.

The general conclusion is that the MSW effect survives the presence of random perturbations at least for small fluctuation values. There are indications that their effect can become important for large fluctuations.

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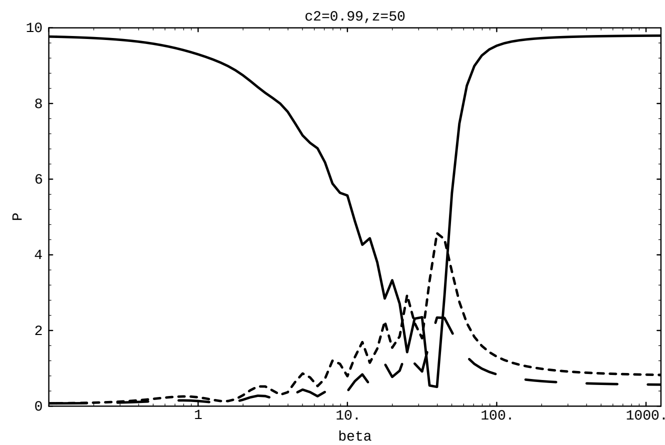


Figure 1: The "total cross sections" ${}^{W}\sigma_1^1$ (short dash) and σ_2 (longer dash) for a neutrino produced at the Sun ($\rho_0/\lambda = 50, \cos^2 = 0.99$) as a function of $\beta = \Delta m^2/2E\lambda$ (see Eqs.(24-25)). The continuos line is the survival probability ($\times 10$).

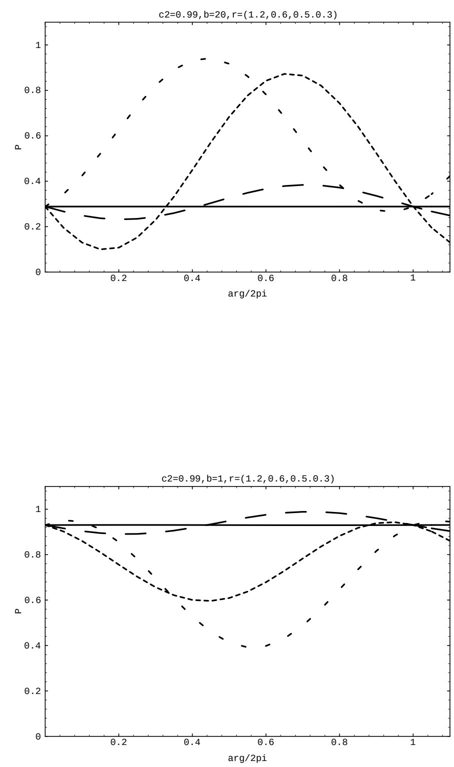


Figure 2: The survival probability as a function of an arbitrary phase shift introduced in different positions along the trajectory of a neutrino created near the Sun core ($\cos^2\theta = 0.99$,$\beta = 1, 20$) . For $\beta = 20$ the resonance is situated at $r/r_0 \approx 0.5{-}0.6$. The continuos line correspond to $r/r_0 = 1.2$: well ahead the resonance region. The longer dashed to $r/r_0 = 0.6$: inside the resonance. The largest variation (both shorter dashed lines) correspond to $r/r_0 = 0.5, 0.3$ at the beginning or clearly before the resonance respectively.

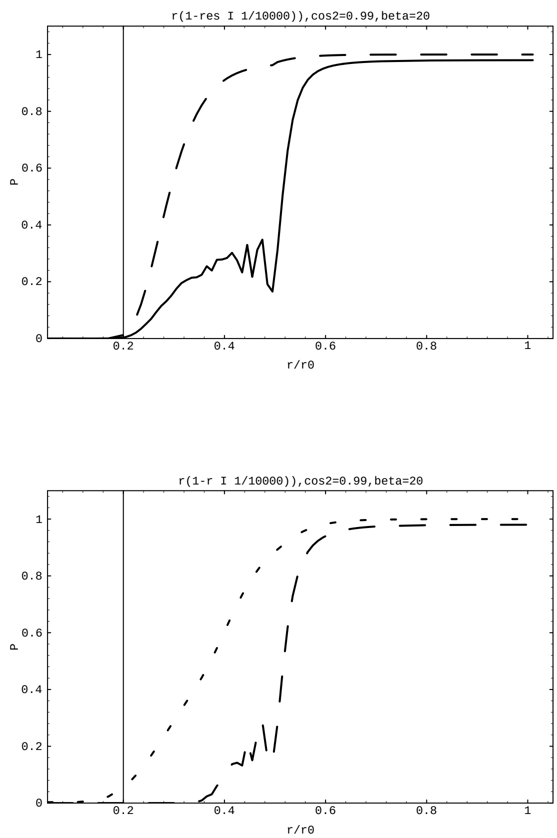


Figure 3: The ν_e coherent survival probability as function of the neutrino creation point. (Top) P_{ee}^{coh} (continuos) and P^{coh} (dashed) computed using Σ^{eff} given by Eq.(82). Bottom, the same probabilities computed using $U_r(\rho_0(1-ik\rho_0/2))$. ($k = 10^{-4}, \beta = 20, \cos^2\theta = 0.99$ for both figures.)

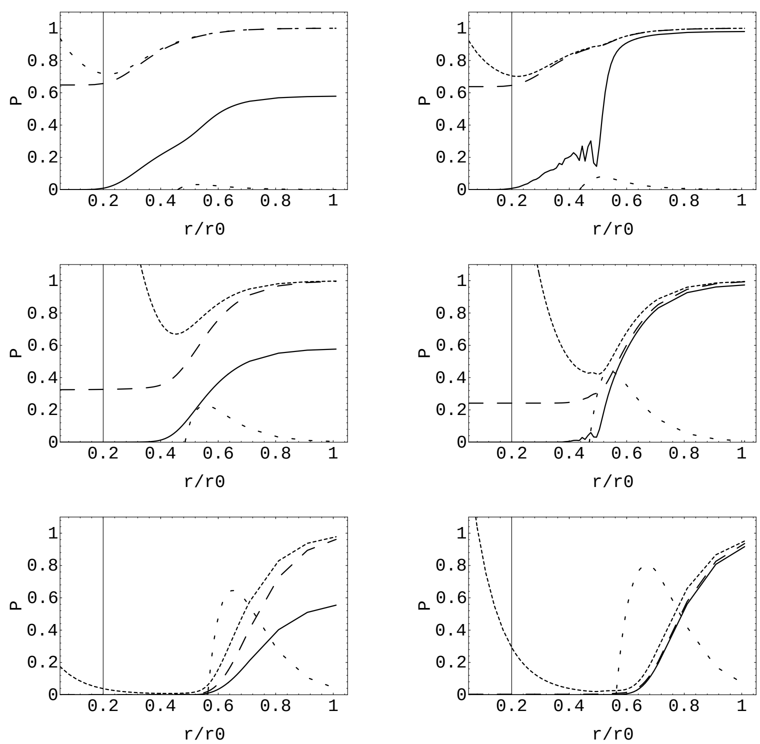


Figure 4: Coherent probabilities P^{coh} (longer dash), P^{coh}_{ee} (continuous line) as a function of k and the mixing angle. They are computed using $U_r(\rho_0(1 - ik\rho_{res}/2))$. $k = 10^{-4}, 10^{-3}, 10^{-2}$ for upper, middle, lower figures respectively. Right figures: large mixing, $\cos^2\theta = 0.70$; left, small mixing $\cos^2\theta = 0.99$. In all the cases $\beta = 20$. It is also depicted P^M (Eq.(72)) (short denser dash) and $C_{min} = 1 - P^{coh} - k\rho^2/2$ (short less denser dash).

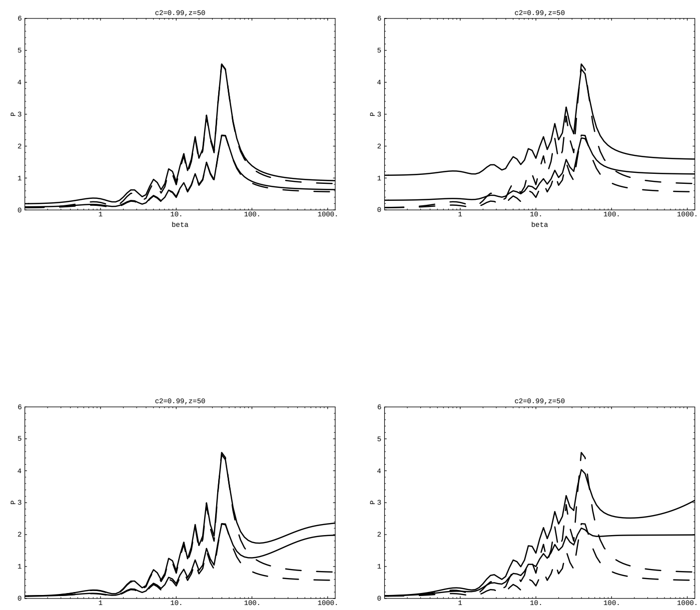
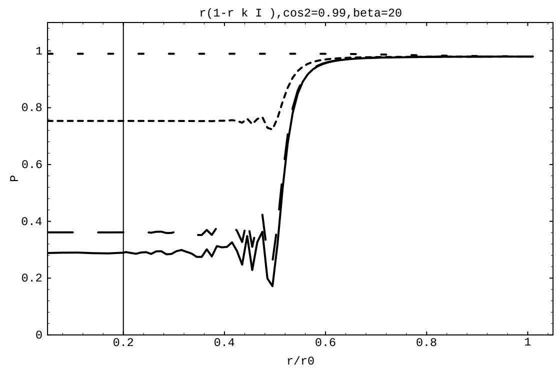


Figure 5: $^{W}\sigma_1^1$ (line) and σ_2 (dashed line) for a neutrino with the same characteristics as in Fig.(1) in presence of a random perturbation. Two top figures: average amplitude computed using Σ^{eff} from Eq.(82). Lower figures: using $U_r(\rho_{or})$. $k = 10^{-4}, 10^{-3}$ ($\approx 1\%, 5\%$ fluctuations) respectively left and right.



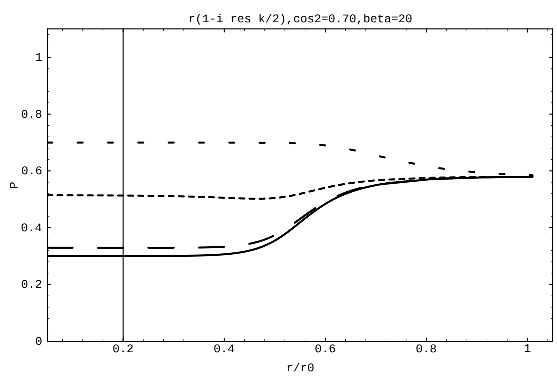


Figure 6: The ν_e survival probability as function of the neutrino creation point computed using the *ansatz* of Section 2.2, Eq.(7) with $\rho_{or} = \rho_0(1 - ik\rho_{res}/2)$. Continuos line: non-random probability ($k = 0$). Dashed lines: $k = 10^{-4}, 10^{-3}, 10^{-2}$ respectively. For both figures $\beta = 20$; $\cos^2\theta = 0.99$ (Top), $\cos^2\theta = 0.70$ (Bottom).